Induced current and Aharonov-Bohm effect in graphene

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The effect of vacuum polarization in the field of an infinitesimally thin solenoid at distances much larger than the radius of solenoid is investigated. The induced charge density and induced current are calculated. Though the induced charge density turned out to be zero, the induced current is a finite periodical function of the magnetic flux Φ. The expression for this function is found exactly in a value of the flux. The induced current is equal to zero at the integer values of Φ/Φ_0 as well as at half-integer values of this ratio, where $\Phi_0 = 2\pi\hbar c/e$ is the elementary magnetic flux. The latter is a consequence of the Furry theorem and periodicity of the induced current with respect to magnetic flux. As an example we consider the graphene in the field of solenoid perpendicular to the plane of a sample.

The Aharonov-Bohm effect,[1] scattering of a charged particle off an infinitesimally thin solenoid, which is absent in classical electrodynamics, has been investigated in numerous papers, see review.[2] Both nonrelativistic[1] and relativistic[3–6] equations have been considered. Similar effects having topological origin have been studied in quantum field theory in Refs. 7 and 8. Intensive investigation of the topological effects in condensed-matter systems has been performed recently both experimentally and theoretically in Refs. 9–12. New possibilities to study topological effects in quantum electrodynamics (QED) have appeared after recent successful fabrication of a monolayer graphite (graphene), see Ref. 13 and recent review.[14] The single-electron dynamics in graphene is described by a massless two-component Dirac equation[15–18] so that graphene represents a peculiar two-dimensional (2D) version of massless QED. This version is essentially simpler than conventional QED because effects of retardation are absent in graphene. However, the "fine structure constant" $\alpha = e^2/\hbar v_F \sim 1$, since the Fermi velocity $v_F \approx 10^6$ m/s $\approx c/300$ (where c is the velocity of light), and therefore we have a strong-coupling version of QED. Below we set $\hbar = c = 1$.

Existence of induced charge density in the electric field of heavy nucleus due to vacuum polarization is one of the most important effects of QED. This problem was investigated in detail in many papers, see, e.g., Refs. 19–22. Charged impurity screening in graphene can also be treated in terms of vacuum polarization.[23–33] In Ref. 32, the induced charge density in graphene has been investigated analytically using convenient integral representation for the Green's function of the two-dimensional Dirac equation of electron in a Coulomb field. Calculation of the induced charge has been performed exactly in the charge of impurity. In Ref. 32, the Green's function has been obtained following the method based on the operator technique suggested in Ref. 34. In the present Brief Report, we use similar integral representation for the Green's function to derive the induced current in the field of infinitesimally thin solenoid. Calculation is performed for arbitrary value of the magnetic flux Φ.

The induced density and induced current in the vector potential

$$A(r) = \frac{\Phi[\boldsymbol{\nu} \times \boldsymbol{r}]}{2\pi r^2},$$

where $\boldsymbol{\nu}$ is the unit vector directed along z axis, have the form

$$\rho_{ind}(\mathbf{r}) = -ieN \int_C \frac{d\epsilon}{2\pi} \text{Tr}\{G(\mathbf{r}, \mathbf{r}|\epsilon)\},$$

$$\boldsymbol{J}_{ind}(\mathbf{r}) = -ieNv_F \int_C \frac{d\epsilon}{2\pi} \text{Tr}\{\boldsymbol{\sigma} G(\mathbf{r}, \mathbf{r}|\epsilon)\} \tag{1}$$

where $N = 4$ reflects the spin and valley degeneracies, and the Green's function $G(\boldsymbol{r}, \boldsymbol{r}'|\epsilon)$ satisfies the equation

$$\{\epsilon - v_F \boldsymbol{\sigma} \cdot [\boldsymbol{p} - eA(\boldsymbol{r})]\}G(\boldsymbol{r}, \boldsymbol{r}'|\epsilon) = \delta(\boldsymbol{r} - \boldsymbol{r}')I. \tag{2}$$

Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$, and σ_i are the Pauli matrices; $\mathbf{p} = (p_x, p_y)$ is the momentum operator, $\boldsymbol{r} = (x, y)$, and $I = \text{diag}\{1, 1\}$. The matrixes $\boldsymbol{\sigma}$ do not act on the spin variables but on the pseudospin ones and the spin degrees of freedom are taken into account in a factor N. According to the Feynman rules, the contour of integration over ϵ goes below the real axis in the left half plane and above the real axis in the right half plane of the complex ϵ plane. It is convenient to write the function $G(\boldsymbol{r}, \boldsymbol{r}'|\epsilon)$ as

$$G(\boldsymbol{r}, \boldsymbol{r}'|\epsilon) = [\epsilon + v_F \boldsymbol{\sigma} \cdot (\boldsymbol{p} - eA)]D(\boldsymbol{r}, \boldsymbol{r}'|\epsilon), \tag{3}$$

where $D(\boldsymbol{r}, \boldsymbol{r}'|\epsilon)$ is the Green's function of the squared Dirac equation,

$$\{\epsilon^2 - v_F^2[\boldsymbol{p} - eA(\boldsymbol{r})]^2 + v_F^2 \Phi \delta(\boldsymbol{r})\sigma_3\}D(\boldsymbol{r}, \boldsymbol{r}'|\epsilon) = \delta(\boldsymbol{r} - \boldsymbol{r}')I. \tag{4}$$

For $r \neq 0$ and $r' \neq 0$, we can omit the term with δ function so that

$$\{\epsilon^2 - v_F^2[\boldsymbol{p} - eA(\boldsymbol{r})]^2\}D(\boldsymbol{r}, \boldsymbol{r}'|\epsilon) = \delta(\boldsymbol{r} - \boldsymbol{r}')I. \tag{5}$$

The Eq. (5) has regular and singular solutions at $r = 0$ and $r' = 0$. The Green's function of the Dirac equation with the magnetic-solenoid field was considered in Ref. 35 taking into

account both regular and singular parts. The singular solutions originate from the singular behavior at $r=0$ of the vector potential $A(r)$. Therefore, to find the correct superposition of regular and singular solutions it is necessary to perform the appropriate regularization. If we take in mind a real solenoid, then the natural regularization is the finite radius R of this solenoid, see Ref. 8. Then it is possible to show that, to calculate the induced charge density and the induced current at $r \gg R$, we can use the regular Green's function of the Eq. (5) while the singular part of the Green's function as well as the term with δ function in Eq. (4) determine these quantities at $r \lesssim R$. The induced current and the induced charge density at $r \lesssim R$ depend on the magnetic field distribution inside the solenoid and, therefore, are model dependent. At $r \gg R$, the contribution of the singular part of the Green's function to the integrand in Eq. (1) contains a factor $(\epsilon R)^\beta$ with some positive β. The main contribution to the integral over ϵ at $r \gg R$ is given by the region $\epsilon \sim 1/r$ so that the contribution of the singular part of the Green's function to the induced current is suppressed by the factor $(R/r)^\beta$. This situation is completely similar to the problem of a finite nuclear size at the calculation of the vacuum polarization effects in heavy atoms, see, e.g., Ref. 36. In quantum field theory it is possible to consider regularizations different from the solenoid radius R. As a result some uncertainty appears in the prediction of physical quantities.[4,7,8] In the present Brief Report we consider the induced charge and induced density at $r \neq 0$ which are model independent. Substituting the function $D(r, r' | \epsilon)$ to Eq. (5) in the form

$$D(r, r' | \epsilon) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} A_m(r, r' | \epsilon) I, \qquad (6)$$

and right-hand side of the Eq. (5) as

$$\delta(r - r') I = \frac{\delta(r - r')}{2\pi\sqrt{rr'}} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} I$$

we obtain that the function $A_m(r, r' | \epsilon)$ satisfies the equations

$$\left(\frac{\epsilon^2}{v_F^2} + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(m-\gamma)^2}{r^2} \right) A_m(r, r' | \epsilon) = \frac{\delta(r - r')}{v_F^2 \sqrt{rr'}}, \quad (7)$$

where $\gamma = e\Phi/(2\pi)$. One can see that the function $A_m(r, r' | \epsilon)$ can be obtained from the Green's function for the free radial Schrödinger equation in the 2D case by the substitution $m \to m - \gamma$. A convenient integral representation for the function $A_m(r, r' | \epsilon)$ can be obtained using the operator method developed in Ref. 34 at the calculation of the Green's function for the Dirac equation of an electron in a Coulomb field in three-dimensional space. This method was recently used in Ref. 32 for the case of 2D space. It follows from the results of Ref. 32 at zero Coulomb field that

$$A_m(r, r' | \epsilon) = -\int_0^{\mu\infty} \frac{ds}{v_F^2 \sinh s}$$

$$\times \exp[iE(r + r')\coth s - i\pi\lambda] J_{2\lambda}\left(\frac{2E\sqrt{rr'}}{\sinh s} \right), \qquad (8)$$

where $E = \epsilon/v_F$, $\lambda = |m - \gamma|$, and $J_{2\lambda}(x)$ is the Bessel function, $\mu = +1$ if $\mathrm{Re}\, E > 0$ and $\mu = -1$ if $\mathrm{Re}\, E < 0$. The sign μ takes into account the analytical properties of the Green's function.

Taking into account the analytical properties of the Green's function, the contour of integration with respect to ϵ can be deformed to coincide with the imaginary axis. After these transformations, we obtain that $\rho_{ind}(r) = 0$ as a result of integration over ϵ. This fact can be easily explained because, due to the Furry theorem, $\rho_{ind}(r)$ should be the odd function of $\gamma = e\Phi/(2\pi)$. However, in this case we would obtain that $\rho_{ind}(r)$ is pseudoscalar that contradicts to the parity conservation of the massless 2D Dirac equation. For $J_{ind}(r)$ we have

$$J_{ind}(r) = -\frac{eNv_F}{\pi^2 r^2}[\boldsymbol{\nu} \times \boldsymbol{r}] \sum_{m=0}^{\infty} (m - \gamma) \int_0^\infty dE \int_0^\infty \frac{ds}{\sinh s}$$

$$\times \exp[-2Er\coth s] I_{2\lambda}\left(\frac{2Er}{\sinh s} \right), \qquad (9)$$

where $I_{2\lambda}(x)$ is the modified Bessel function of the first kind. We note that $J_{ind}(r)$, Eq. (9), is an odd function of γ, in accordance with the Furry theorem. To have a possibility to change the order of summation and integration, we introduce some quantity $\delta \ll 1$ as a lower limit of integration over s. After that we take the integral over E and obtain

$$J_{ind}(r) = -\frac{eNv_F}{2\pi^2 r^3}[\boldsymbol{\nu} \times \boldsymbol{r}] \sum_{m=-\infty}^{\infty} (m - \gamma)$$

$$\times \int_\delta^\infty \frac{ds}{\sinh s} \exp[-2\lambda s]. \qquad (10)$$

As should be, $J_{ind}(r)$ depends only on the fractional part $\tilde{\gamma}$ of γ, $|\tilde{\gamma}| < 1$. The quantity $\tilde{\gamma}$ is $\tilde{\gamma} = \gamma - n$ for $\gamma > 0$ and $\tilde{\gamma} = \gamma + n$ for $\gamma < 0$, where n is a maximal integer number less than $|\gamma|$. Then we perform summation over m and set $\delta = 0$. We have

$$J_{ind}(r) = \frac{eNv_F}{2\pi^2 r^3}[\boldsymbol{\nu} \times \boldsymbol{r}] \int_0^\infty \frac{ds}{\sinh s}\left[\tilde{\gamma} \exp(-2|\tilde{\gamma}|s) \right.$$

$$\left. + \tilde{\gamma} \exp(-s)\frac{\cosh(2\tilde{\gamma}s)}{\sinh s} - \frac{\sinh(2\tilde{\gamma}s)}{2\sinh^2 s} \right]. \qquad (11)$$

Taking the integral over s we finally arrive at

$$J_{ind}(r) = \frac{eNv_F}{16\pi} F(\tilde{\gamma}) \mathrm{curl}\left(\frac{\boldsymbol{\nu}}{r} \right),$$

$$F(\tilde{\gamma}) = (1 - 2|\tilde{\gamma}|)^2 \tan(\pi\tilde{\gamma}). \qquad (12)$$

It is interesting that $J_{ind}(r)$ equals to zero at $|\tilde{\gamma}| = 1/2$, Fig. 1. This may be explained as follows. Due to invariance of
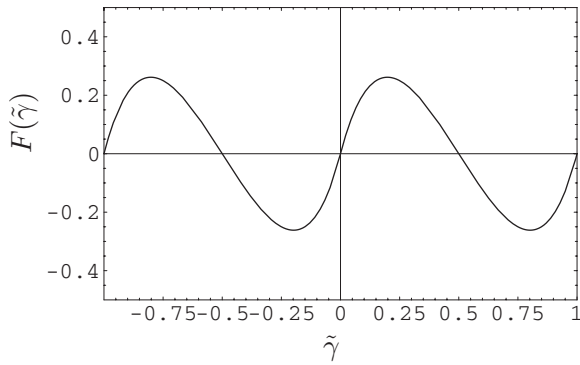
FIG. 1. Dependence of the function $F(\tilde{\gamma})$, Eq. (12), on the fractional part $\tilde{\gamma}$ of $\gamma = e\Phi/(2\pi)$, where Φ is the magnetic flux.

$J_{ind}(r)$ under the substitution $\gamma \to \gamma - 1$ we have $F(|\tilde{\gamma}|) = F(|\tilde{\gamma}| - 1)$, and due to the Furry theorem it should be $F(\tilde{\gamma}) = -F(-\tilde{\gamma})$. From these two relations we obtain that $F(\pm 1/2) = 0$.

The induced charge density and current in the presence of an infinitesimally thin solenoid were also considered in Refs. 37 and 38. The results of these papers contain the contribution of both singular and regular parts of the Green's function since a regularization for the field was not applied. Therefore, the expressions for the induced charge density and current contain uncertainty which does not allow one to make any explicit predictions for these quantities. Note that our results are in agreement with the contribution of the regular part of the Green's function in Ref. 38 [first term in Eq. (6.14)].

To summarize, we have investigated the effect of vacuum polarization in the field of an infinitesimally thin solenoid at distances much larger than the radius of solenoid. It turns out that the induced charge density is zero. We have derived exactly in a magnetic flux the expression for the induced current. This current is a periodic function of the magnetic flux and is equal to zero not only at the integer values of Φ/Φ_0 but also at half-integer values of this ratio. Though the system considered in our Brief Report consists of graphene and a solenoid perpendicular to the plane of a sample, the results can be easily generalized for another systems such as studied in Ref. 9.

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